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Continuos g-contrácticles y la relación que guarda con otras estructuras topológicas.

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POR ARTÍCULOS ESPECIALIZADOS

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Resumen

Un continuo es un espacio métrico, compacto, conexo y no vacío. Un continuo es llamado contráctil si existe una función continua $H : X \times I \to X$ tal que $H(x, 0) = x$ y $H(x, 1) = p$ para algún $p \in X$. La contractibilidad de un continuo es una propiedad que ha sido ampliamente estudiada y de la que se pueden deducir propiedades topológicas interesantes como la unicoherencia y la arco-conexidad. Dos generalizaciones naturales del concepto de contractibilidad son la g-contractibilidad y la pseudo-contractibilidad, dichas nociones son objeto de investigación de diversos autores. En este proyecto de investigación se da una generalización de los conceptos antes mencionados la cual es llamada *g-pseudo-contractibilidad*. Al ser un nuevo concepto dentro de la Tería de Continuos, en este trabajo se presentan propiedades generales de dicho concepto. Además se dan respuestas parciales a la siguiente pregunta ¿bajo qué condiciones un continuo es o no es g-pseudo-contráctil?.

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On g-pseudo-contractibility of continua

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ABSTRACT

Let f and g be maps between topological spaces X and Y. The maps f and g are called *pseudo-homotopic* provided that there exist a continuum C, points $a, b \in C$ and a map $H : X \times C \to Y$ such that $H(x, a) = g(x)$ and $H(x, b) = f(x)$; the map H is called a *pseudo-homotopy between f and g*. A topological space X is said to be g-pseudo-contractible provided that there exists a pseudo-homotopy between an onto map from X to X and a constant map. The main purpose of this paper is to present the concept of g-pseudo-contractibility that generalizes the notions of g-contractibility and pseudo-contractibility showing general facts about it.

1. Introduction

A *continuum* is a nonempty compact connected metric space. A continuum is *contractible* provided that the identity map is homotopic to a constant map. Contractibility is one of the most important topological properties in general topology and it has been considered by several authors. Recently, two natural generalizations of this concept have been studied, namely, g-contractibility and pseudo-contractibility.

A continuum X is *g-contractible* provided that there exists an onto map f from X to X such that f is homotopic to a constant map. Notice that each contractible continuum is g-contractible but the converse of this statement fails. D. Bellamy in [1] defined g-contractibility mainly to study which continua are both continuous image and continuous preimage of the Cantor fan. The interested reader is referred to [1], [3], [8], [14].

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On the other hand, R. H. Bing introduced the concept of pseudo-contractibility. A continuum is *pseudo-contractible* provided that its identity map is pseudo-homotopic to a constant map. Notice that all contractible continuum is pseudo-contractible. W. Kuperberg gave the first example to show that notions of contractibility and pseudo-contractibility are different. Interesting properties of this concept are presented in [2], [4], [5], [6], [9], [11] and [15].

Now, in connection with both concepts, in this paper, we introduce the notion of g-pseudo-contractibility. A continuum X is *g-pseudo-contractible* if there exists an onto map from X to X pseudo-homotopic to a constant map. Every g-contractible continuum is g-pseudo-contractible and every pseudo-contractible continuum is g-pseudo-contractible. So, every contractible continuum is g-pseudo-contractible. The aim of this paper is to present general results related to g-pseudo-contractibility.

Finally, in [4] the concept of *pseudo-contractibility with respect to* is introduced. The authors gave general facts related to this concept and it is used to get some results concerning pseudo-contractibility and other properties. Here, we bring to light the concept g-pseudo-contractibility with respect to and we give general facts about it.

The current paper is divided in five sections. After Introduction and Preliminaries, in Section 3, we present aspects derived from the g-pseudo-contractibility. Some of our results are that being g-pseudo-contractible continuum does not imply neither being pseudo-contractible continuum nor being g-contractible continuum, the g-pseudo-contractibility is a productive property but we show a non-g-pseudo-contractible continuum W such that $W \times I$ is g-pseudo-contractible. Section 4 is dedicated to extend the notion of pseudo-contractibility with respect to and general results are proved. Finally, in Section 5, we give examples to see that the g-pseudo-contractibility is not invariant under the following classes of maps: open, monotone, light and retractions.

2. Preliminaries

We denote by $\mathbb{N}$ the set of all positive integers. A *map* means a continuous function between topological spaces. The interval $[0,1]$ is denoted by I. Let f and g be maps between topological spaces X and Y. The maps f and g are called *pseudo-homotopic* provided that there exist a continuum C, points $a, b \in C$ and a map $H : X \times C \to Y$ such that $H(x,a) = g(x)$ and $H(x,b) = f(x)$; in this case the map H is called a *pseudo-homotopy between f and g*. If there exits a map $H : X \times I \to Y$ satisfying that $H(x,0) = f(x)$ and $H(x,1) = g(x)$, then we say that f and g are *homotopic*, and the map H is called a *homotopy between f and g*. Each map pseudo-homotopic (homotopic) to a constant map is called *null-pseudo-homotopic* (*null-homotopic*).

3. Main results

A continuum X is said to be:

- *Contractible* if the identity map on X is null-homotopic.
- *Pseudo-contractible* if the identity map on X is null-pseudo-homotopic.
- *g-contractible* provided that there exists a null-homotopic map from X onto X.
- *g-pseudo-contractible* provided that there exists a null-pseudo-homotopic map from X onto X.

Into the following result is presented all possible relationships among these concepts.

Theorem 3.1. *Let X be a continuum.*

1. *If X is contractible, then X is g-contractible.*

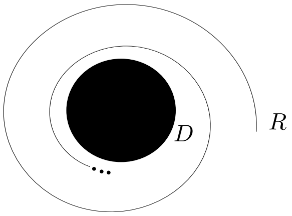

Fig. 1. Continuum Y of Example 3.3.

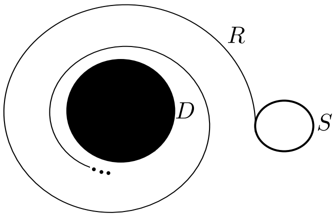

Fig. 2. Continuum X of Example 3.4.

2. *If X is contractible, then X is pseudo-contractible.*
3. *If X is g-contractible, then X is g-pseudo-contractible.*
4. *If X is pseudo-contractible, then X is g-pseudo-contractible.*

The following examples show that the concepts are not equivalent.

Example 3.2. By [3, Remark 2.15, p. 464], each locally connected continuum is g-contractible. On the other hand, [10, Lemma 19.4, p. 156] and [10, Lemma 19.7, p. 158] together imply that each contractible continuum is unicoherent (unicoherent means that the intersection of each two of its subcontinua whose union is the continuum is connected). Thus, we can conclude that the unit circle S^1 is g-contractible but it is not contractible. This proves that the converse of (1) in Theorem 3.1 fails.

The continuum described below is the classical example to show that the converse of (2) of Theorem 3.1 does not hold. We repeat it here to see that the converse of (3) of Theorem 3.1 is not true too.

Example 3.3. In the complex plane $\mathbb{C}$, let $R = \{\frac{t+2}{t+1}e^{it} : t \in [0, \infty)\}$ be a spiral approaching to the unit circle and let $D = \{re^{it} : r \in [0, 1], t \in [0, 2\pi]\}$ be the unit disc. Set $Y = R \cup D$ (see Fig. 1). W. Kuperberg showed that the continuum Y is pseudo-contractible but it is not contractible. On the other hand, it is well known that every g-contractible space is arcwise connected (see [3, Remark 2.10, p. 463]). Since Y is not arcwise-connected, it can not be g-contractible. From (4) of Theorem 3.1, it follows that Y is g-pseudo-contractible. This implies that the converse of (3) of Theorem 3.1 fails.

We now prove that the converse of (4) in Theorem 3.1 fails.

Example 3.4. In the complex plane $\mathbb{C}$, let $Y = R \cup D$ be the continuum described in Example 3.3, we define $S^+ = \{3 + e^{it} : t \in [0, \pi]\}$ and $S^- = \{3 + e^{it} : t \in [\pi, 2\pi]\}$. Let $S = S^+ \cup S^-$ be the unit circle centered in $3 + 0i$. Set $X = Y \cup S$ (see Fig. 2). Notice that X is not unicoherent. From [4, Corollary 56, p. 68], the continuum X is not pseudo-contractible. In order to prove that X is g-pseudo-contractible, let $f : X \to X$ be defined by:

$$f(x) = \begin{cases} x, & \text{if } x \in Y, \\ 3 + e^{i(2t-3\pi)}, & \text{if } x = 3 + e^{it} \in S^{+}, \\ 3 + e^{i(-2t+\pi)}, & \text{if } x = 3 + e^{it} \in S^{-}. \end{cases}$$

Let us show that f is onto. Let $x \in X$. By definition of f It enough to assume that $x \in S$. Then, there exists $r \in [-\pi, \pi]$ such that $x = 3 + e^{ir}$. Observe that $t = \frac{\pi + r}{2} \in [0, \pi]$ and $f(3 + e^{it}) = x$.

Now, we shall prove that f is null-pseudo-homotopic. Let $E = \{e^{it} : t \in [0, 2\pi]\} \cup \{re^{i0} : 0 \leq r \leq 1\}$ be the union of the unit circle and one of its radius, and let $J = \{re^{i0} : 2 \leq r \leq 3\}$ be the convex segment between $2 + 0i$ and $3 + 0i$ in $\mathbb{C}$. Set $F = R \cup E$ and $C = F \cup J$. Notice that the space C is a continuum.

The function $H : X \times C \to X$ defined by:

$$H(x,c) = \begin{cases} \frac{t+s+2}{t+s+1} e^{i(t+s)}, & \text{if } (x,c) = (\frac{t+2}{t+1} e^{it}, \frac{s+2}{s+1} e^{is}) \in R \times R, \\ max\{||x||, ||c||, 1\}^{-1} \cdot xc, & \text{if } (x,c) \in (D \times F) \cup (Y \times E), \\ c, & \text{if } (x,c) \in S \times F, \\ x, & \text{if } (x,c) \in Y \times J, \\ 3 + e^{i[(r-2)(2t-3\pi)]}, & \text{if } (x,c) = (3 + e^{it}, re^{i0}) \in S^{+} \times J, \\ 3 + e^{i[(r-2)(-2t+\pi)]}, & \text{if } (x,c) = (3 + e^{it}, re^{i0}) \in S^{-} \times J, \end{cases}$$

is a pseudo-homotopy between f and a constant mapping.

The following remarks follow immediately from the definition of g-contractibility and g-pseudo-contractibility. We will use them along of this paper without mentioning them explicitly.

Remark 3.5. A continuum X is g-contractible if and only if there exists a map $H : X \times I \to X$ such that $H(X \times \{0\}) = X$ and $H(X \times \{1\}) = \{p\}$ for some $p \in X$.

Remark 3.6. A continuum X is g-pseudo-contractible if and only if there exist a continuum C, points $a, b \in C$ and a map $H : X \times C \to X$ such that $H(X \times \{a\}) = X$ and $H(X \times \{b\}) = \{p\}$ for some $p \in X$.

Theorem 3.7. *Let X be a continuum. Then X is g-pseudo-contractible if and only if there exist a continuum C, points $a, b \in C$ and a map $f : X \times C/(X \times \{b\}) \to X$ such that $f(\pi(X \times \{a\})) = X$, where π denotes the quotient map from $X \times C$ to $X \times C/(X \times \{b\})$.*

Proof. Suppose that X is g-pseudo-contractible. Then, there exist a continuum C, points $a, b \in C$ and a map $H : X \times C \to X$ such that $H(X \times \{a\}) = X$ and $H(X \times \{b\})$ is a singleton point of X. The Transgression Theorem [7, Theorem 3.2, p. 123] ensures that the composition $H \circ \pi^{-1}$ is a map. We define $f : X \times C/(X \times \{b\}) \to X$ by $f = H \circ \pi^{-1}$. Thus, $f(\pi(X \times \{a\})) = H \circ \pi^{-1}(\pi(X \times \{a\})) = H(X \times \{a\}) = X$.

Now, we assume that there exist a continuum C, points $a, b \in C$ and a map $f : X \times C/(X \times \{b\}) \to X$ such that $f(\pi(X \times \{a\})) = X$. Let $H : X \times C \to X$ be the map defined by $H(x, c) = (f \circ \pi)(x, c)$. Thus, $H(X \times \{a\}) = f(\pi(X \times \{a\})) = X$ and $H(X \times \{b\}) = f(\pi(X \times \{b\}))$ is a singleton point in X. Therefore, X is g-pseudo-contractible. □

Recall that the quotient space $X \times I/(X \times \{1\})$ is denoted by $Cone(X)$ and the subset $X \times \{0\}$ of $Cone(X)$, denoted by $B(X)$, is called the base of the cone. Similar arguments used in the proof of Theorem 3.7 can be used to show the following result. Compare with [3, Remark 2.10, p. 463].

Theorem 3.8. *Let X be a continuum. Then, X is g-contractible if and only if there exists an onto map $f : Cone(X) \to X$ such that $f(B(X)) = X$.*

Notation 3.9. Given a continuum C and points $a, b \in C$, we will say that X is g-pseudo-contractible with factor space $C(a, b)$ if there exists a map $H : X \times C \to Y$ such that $H(X \times \{a\}) = X$ and $H(X \times \{b\}) = \{p\}$ for some $p \in X$. If there is not confusion, we say that X is g-pseudo-contractible with factor space C.

Lemma 3.10. *Let X, C and D be continua. If X is g-pseudo-contractible with factor space* $\mathrm{C(a, b)}$ *and there exists a map $f : D \to C$ such that $a, b \in f(D)$, then X is g-pseudo-contractible with factor space D.*

Proof. By our assumption, there exists a map $H : X \times C \to X$ such that $H(X \times \{a\}) = X$ and $H(X \times \{b\}) = \{p\}$ for some $p \in X$. Now, there exist $c, d \in D$ such that $f(c) = a$ and $f(d) = b$. Let $G : X \times D \to X$ be the map defined by $G(x, t) = H(x, f(t))$. Observe that $G(X \times \{c\}) = H(X \times \{f(c)\}) = H(X \times \{a\}) = X$ and $G(X \times \{d\}) = H(X \times \{f(d)\}) = H(X \times \{b\}) = \{p\}$. In conclusion, X is g-pseudo-contractible with factor space D. □

The next results follow immediately from Lemma 3.10.

Theorem 3.11. *Let X, C and D be continua. If X is g-pseudo-contractible with factor space C and C is continuous image of D, then X is g-pseudo-contractible with factor space D.*

Theorem 3.12. *Let X and C be continua. If X is g-pseudo-contractible with factor space* $\mathrm{C(a, b)}$ *and D is a subcontinuum of C containing a and b, then X is g-pseudo-contractible with factor D. Moreover, X is g-pseudo-contractible with factor $I(a, b)$, where $I(a, b)$ is an irreducible continuum between a and b.*

Theorem 3.13. *If X is g-pseudo-contractible with factor space $C(a, b)$ and there exists an arc component of C containing a and b, then X is g-contractible.*

Corollary 3.14. *If X is g-pseudo-contractible with factor C and C is arcwise connected, then X is g-contractible.*

The Urysohn's Lemma ensures that the unit interval is continuous image of any continuum. Thus, we obtain the following result.

Theorem 3.15. *Let X be a continuum. If X is g-contractible, then X is g-pseudo-contractible with any factor space.*

Two continua X and Y are said to be *continuously equivalent* if there are two onto maps $\mu : X \to Y$ and $\rho : Y \to X$.

The following results can be compared with [4, Corollary 15, p. 61] and [3, Remark 2.15, p. 464], respectively.

Theorem 3.16. *Let X, C and D be continua such that C is continuously equivalent to D. Then, X is g-pseudo-contractible with factor space C if and only if X is g-pseudo-contractible with factor space D.*

Theorem 3.17. *Let X and Y be continua such that X is continuously equivalent to Y. Then X is g-pseudo-contractible if and only if Y is g-pseudo-contractible.*

Proof. By hypothesis, there exist onto maps $\alpha : X \to Y$ and $\gamma : Y \to X$. Suppose that X is g-pseudo-contractible. Then, there exist a continuum C, points $a, b \in C$ and a map $H : X \times C \to X$ satisfying

$H(X \times \{a\}) = X$ and $H(X \times \{b\}) = \{p\}$ for some $p \in X$. Now, let $G : Y \times C \to Y$ be the map defined by $G(y,t) = \alpha(H(\gamma(y),t))$. Observe that $G(Y \times \{a\}) = \alpha(H(\gamma(Y) \times \{a\})) = \alpha(H(X \times \{a\})) = \alpha(X) = Y$ and $G(Y \times \{c\}) = \alpha(H(\gamma(Y) \times \{c\})) = \alpha(H(X \times \{c\})) = \{\alpha(p)\}$. Therefore, Y is g-pseudo-contractible. □

Observe that I is continuously equivalent to the unit circle S^1. The continuum I is pseudo-contractible (contractible) but the continuum S^1 is not pseudo-contractible (is not contractible) (see [4, Corollary 58, p. 69]). This shows that the last theorem fails when we consider contractibility or pseudo-contractibility instead of g-pseudo-contractibility.

Concerning to products we have that the g-pseudo-contractibility of continua is a productive property. Compare with [3, Theorem 3.1, p. 464] and [4, Corollary 22, p. 62]

Theorem 3.18. *If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of g-pseudo-contractible continua, then $\prod_{n\in\mathbb{N}} X_n$ is g-pseudo-contractible.*

Proof. By hypothesis, for each $n \in \mathbb{N}$, there exist a continuum C_n, points a_n, $b_n \in C_n$ and a map $H_n : X_n \times C_n \to X_n$ such that $H_n(X_n \times \{a_n\}) = X_n$ and $H_n(X_n \times \{b_n\}) = \{p_n\}$ for some $p_n \in X_n$. Since C_n is a continuum for each $n \in \mathbb{N}$, we have that $\prod_{n\in\mathbb{N}} C_n$ is a continuum. Now, we define $H : \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} C_n \to \prod_{n\in\mathbb{N}} X_n$ by $H((x_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}) = (H_n(x_n, c_n))_{n\in\mathbb{N}}$. The map H satisfies that $H(\prod_{n\in\mathbb{N}} X_n \times \{(a_n)_{n\in\mathbb{N}}\}) = \prod_{n\in\mathbb{N}} (H_n(X_n \times \{a_n\}))_{n\in\mathbb{N}} = \prod_{n\in\mathbb{N}} X_n$ and $H(\prod_{n\in\mathbb{N}} X_n \times \{(b_n)_{n\in\mathbb{N}}\}) = \prod_{n\in\mathbb{N}} (H_n(X_n \times \{b_n\}))_{n\in\mathbb{N}} = \{(p_n)_{n\in\mathbb{N}}\}$. Therefore, $\prod_{n\in\mathbb{N}} X_n$ is g-pseudo-contractible. □

In Section 5, we show that there exists a non g-pseudo-contractible continuum W such that $W \times I$ is g-pseudo-contractible. Thus, the converse of Theorem 3.18 is not true.

Corollary 3.19. *If X is a g-pseudo-contractible continuum, then*

1. *X^n is g-pseudo-contractible for each $n \in \mathbb{N}$.*
2. *$X \times I$ is g-pseudo-contractible.*
3. *$\prod_{n\in\mathbb{N}} X_n$ is g-pseudo-contractible where $X_n = X$ for each $n \in \mathbb{N}$.*

The following lemma will be used in the next theorem.

Lemma 3.20. *Let X be a continuum and let $B_1, \dots, B_k$ be subcontinua of X such that $X = \bigcup_{i=1}^{k} B_i$. If for each $i \in \{1, \dots, k\}$ there exist maps $H_i : X \times I \to B_i$ and $G_i : X \times I \to B_i$ satisfying $H_i(X \times \{0\}) = B_i = G_i(X \times \{0\})$ and for each $i \leq k-1$, $G_i(x, 1) = H_{i+1}(x, 1)$ for each $x \in X$ and $G_k(X \times \{1\})$ is a singleton of X, then there exists an onto map $G : Cone(X) \to X$.*

Proof. We define $F : X \times I \to X$ by

$$F(x,t) = \begin{cases} H_i(x, 2(kt + 1 - i)), & \text{if } t \in [\frac{2i-2}{2k}, \frac{2i-1}{2k}],\ i \leq k, \\ G_i(x, 2(i - kt)), & \text{if } t \in [\frac{2i-1}{2k}, \frac{2i}{2k}],\ i \leq k. \end{cases}$$

Notice that for each $i \in \{1, \dots, k-1\}$,

$$H_i\left(x, 2\left(k\left(\frac{2i-1}{2k}\right) + 1 - i\right)\right) = G_i\left(x, 2\left(i - k\left(\frac{2i-1}{2k}\right)\right)\right)$$

and

$$G_i\left(x, 2\left(i - k\left(\frac{2i}{2k}\right)\right)\right) = H_{i+1}\left(x, 2\left(k\left(\frac{2i-2}{2k}\right) + 1 - i\right)\right).$$

Hence F is well defined. Because for each $i \in \{1, \ldots, k\}$, H_i and G_i are onto maps and $X = \bigcup_{i=1}^{k} B_i$, F is an onto map. Finally, let $G : Cone(X) \to X$ be a map defined by $G = F \circ \pi^{-1}$, where π denotes the quotient map from $X \times I$ to $Cone(X)$. □

A condition under which the product of a continuum X and the interval I is g-contractible is exhibited in the next result by showing $X \times I$ can be g-contractible although X is not.

Theorem 3.21. *Let X be a continuum. If there exist g-contractible subcontinua $B_1, \ldots B_k$ of X such that each one of them is continuous image of X and $X = \bigcup_{i=1}^{k} B_i$, then $X \times I$ is g-contractible.*

Proof. Since $X = \bigcup_{i=1}^{k} B_i$, for each $i \in \{1, \ldots, k-1\}$ we can suppose that $B_i \cap B_{i+1} \neq \emptyset$, let $b_i \in B_i \cap B_{i+1}$. Let $i \in \{1, \ldots, k\}$. The fact that B_i is g-contractible implies that there exist onto maps F_i and T_i from $B_i \times I$ onto B_i such that $F_i(X \times \{0\}) = B_i = T_i(X \times \{0\})$ and $F_i(x, 1) = b_i$, $T_i(x, 1) = b_{i-1}$ for each $x \in B_i$ (see [3, Theorems 2.11 and 2.13, p. 463]). Let $f_i : X \to B_i$ be onto maps. We define $G_i : X \times I \to B_i$ by $G_i(x, t) = F_i(f_i(x), t)$ and $H_i : X \times I \to B_i$ by $H_i(x, t) = T_i(f_i(x), t)$. Thus, $G_i(X \times \{0\}) = B_i = H_i(X \times \{0\})$ and $G_i(x, 1) = b_i$, $H_i(x, 1) = b_{i-1}$ for each $x \in X$. Thus, from Lemma 3.20, there exists an onto map $G : Cone(X) \to X$. Applying [3, Corollary 3.7, p. 465], we conclude that $X \times I$ is g-contractible. □

Observe that the condition: B is continuously equivalent to C implies that $A \times B$ is continuously equivalent to $A \times C$. Hence, the next result follows from the fact I is continuously equivalent to each locally connected continuum (see [13, Theorem 8.18, p. 128]).

Corollary 3.22. *Let X be a continuum and let L be a locally connected continuum. If there exist g-contractible subcontinua $B_1, \ldots B_k$ of X such that each one of them is continuous image of X and $X = \bigcup_{i=1}^{k} B_i$, then $X \times L$ is g-contractible.*

We do not know if Theorem 3.21 and Corollary 3.22 can be extended to g-pseudo-contractibility when the factor spaces are not continuous image of the interval I.

4. g-pseudo-contractibility with respect to

Let X, Y and Z be topological spaces and let $\alpha : X \to X$ and $\gamma : Y \to Y$ be onto maps. We say that:

- the map γ *l-p-nulls Z (l-nulls Z)* provided that for each map $f : Z \to Y$, $\gamma \circ f$ is null-pseudo-homotopic (null-homotopic),
- the map α *r-p-nulls Z (r-nulls Z)* provided that for each map $f : X \to Z$, $f \circ \alpha$ is null-pseudo-homotopic (null-homotopic),
- the ordered pair (α, γ) *p-nulls (nulls)* provided that for each map $f : X \to Y$, $\gamma \circ f \circ \alpha$ is null-pseudo-homotopic (null-homotopic).

The space X is called *left g-pseudo-contractible with respect to* Y *(left g-contractible with respect to* Y*)* if there exists an onto map $\gamma : Y \to Y$ such that γ l-p-nulls X (l-nulls X). We say that X is *right g-pseudo-contractible with respect to* Y *(right g-contractible with respect to* Y*)* if there exists an onto map $\alpha : X \to X$ such that α r-p-nulls Y (r-nulls Y). If there exist onto maps $\alpha : X \to X$ and $\gamma : Y \to Y$ such that (α, γ) p-nulls (nulls), then X is said to be *g-pseudo-contractible with respect to* Y *(*X *is g-contractible with respect to* Y*)*.

Let X and Y be spaces. We say that X is *pseudo-contractible (contractible) with respect to* Y provided that every map $f : X \to Y$ is null-pseudo-homotopic (null-homotopic). Thus, X is pseudo-contractible (contractible) with respect to Y if and only if (id_X, id_Y) p-nulls (nulls).

Proposition 4.1. *Let* X *be a continuum. Then,* X *is g-pseudo-contractible (g-contractible) if and only if* X *is g-pseudo-contractible (g-contractible) with respect to itself.*

Observe that each null-homotopic map is null-pseudo-homotopic. Thus, the following result holds.

Theorem 4.2. *Let* X *and* Y *be continua. If* X *is contractible with respect to* Y*, then* X *is g-pseudo-contractible with respect to* Y*.*

The following lemma will be used in the proof of the most important results of this section, without mention it explicitly.

Lemma 4.3. *Let* X, Y, Z *and* W *be topological spaces and let* $f : X \to Y$*,* $g : Z \to X$ *and* $h : Y \to W$ *be maps. If* f *is null-pseudo-homotopic, then* $h \circ f$*,* $f \circ g$ *and* $h \circ f \circ g$ *are null-pseudo-homotopic.*

Proof. Since f is null-pseudo-homotopic, there exist a continuum C, points $a, b \in C$ and a map $H : X \times C \to Y$ satisfying $H(x, a) = f(x)$ and $H(x, b) = p$ for some $p \in Y$. Defining the maps $F : X \times C \to W$, $G : Z \times C \to Y$ and $T : Z \times C \to W$ by $F(x, t) = h(H(x, t))$, $G(z, t) = H(g(z), t)$ and $T(z, t) = h(H(g(z), t))$, respectively, we can see that $F(x, a) = (h \circ f)(x)$, $F(x, b) = h(p)$, $G(z, a) = (f \circ g)(z)$, $G(z, b) = p$, $T(z, a) = (h \circ f \circ g)(z)$ and $T(z, b) = h(p)$. Thus, $h \circ f$, $f \circ g$ and $h \circ f \circ g$ are null-pseudo-homotopic. □

As a consequence of Lemma 4.3, we have the following results.

Theorem 4.4. *Let* X *and* Y *be continua. If* X *is either right g-pseudo-contractible or left g-pseudo-contractible with respect to* Y*, then* X *is g-pseudo-contractible with respect to* Y*.*

Proof. Assume that X is right g-pseudo-contractible. Let f be an onto map. By hypothesis, there exists an onto map $\alpha : X \to X$ such that α r-p-nulls Y. Thus, $f \circ \alpha$ is null-pseudo-homotopic. Applying Lemma 4.3, we conclude that (α, id_Y) p-nulls. Therefore, X is g-pseudo-contractible with respect to Y. □

Corollary 4.5. *Let* X *be a continuum. If* X *is either right g-pseudo-contractible or left g-pseudo-contractible with respect to itself, then* X *is g-pseudo-contractible.*

Theorem 4.6. *Let* X *be a continuum. The following statements are equivalent:*

1) X *is g-pseudo-contractible (g-contractible).*
2) X *is right g-pseudo-contractible (right g-contractible) with respect to* Y *for each continuum* Y*.*
3) Z *is left g-pseudo-contractible (left g-contractible) with respect to* X *for each continuum* Z*.*
4) X *is g-pseudo-contractible (g-contractible) with respect to* Y *for each continuum* Y*.*
5) Z *is g-pseudo-contractible (g-contractible) with respect to* X *for each continuum* Z*.*

Proof. First, we shall prove that (1) implies both (2) and (3). Assume that X is g-pseudo-contractible. Thus, there exist a null-pseudo-homotopic onto map $g : X \to X$. In order to show (2), let Y be a continuum and let $f : X \to Y$ be a map. Then, the map $f \circ g$ is null-pseudo-homotopic. Thus, g r-p-nulls Y. Now, let us prove (3). Let Z be a continuum and let $h : Z \to X$ be a map. Then, the map $g \circ h$ is null-pseudo-homotopic. We conclude that g l-p-nulls Z. On the other hand, we apply Theorem 4.4 to conclude that (4) is implied by (2) and (5) is implied by (3). Finally, by Proposition 4.1 taking $Y = X = Z$, each one of the statements (4) and (5) implies (1). □

Corollary 4.7. *Let X be a continuum. If X is right g-contractible (left g-contractible) with respect to itself, then X is g-contractible.*

Theorem 4.8. *Let X, Y and D be continua. If X is right g-pseudo-contractible (right g-contractible) with respect to Y and D is continuously equivalent to X, then D is right g-pseudo-contractible (right g-contractible) with respect to Y.*

Proof. Let $\alpha : X \to X$, $\mu : X \to D$ and $\rho : D \to X$ be onto maps such that α r-p-nulls Y. We claim that the map $\mu \circ \alpha \circ \rho$ from D onto D r-p-nulls Y. To see this, let $f : D \to Y$ be a map. Because the map $f \circ \mu$ is defined from X to Y, the map $(f \circ \mu) \circ \alpha$ is null-pseudo-homotopic. From this, it follows that $f \circ \mu \circ \alpha \circ \rho$ is null-pseudo-homotopic. This finishes the proof. □

Corollary 4.9. *Let X and Y be continua. If X and Y are continuously equivalent and X is right g-pseudo-contractible (right g-contractible) with respect to Y, then X and Y are g-pseudo-contractible (g-contractible).*

Theorem 4.10. *Let X, Y and D be continua. If X is left g-pseudo-contractible (left g-contractible) with respect to Y and D is continuously equivalent to Y, then X is left g-pseudo-contractible (left g-contractible) with respect to D.*

Proof. Assume that $\gamma : Y \to Y$ is an onto map in such a way γ l-p-nulls X. By hypothesis, there exist onto maps $\mu : Y \to D$ and $\rho : D \to Y$. We shall prove that the map $\mu \circ \gamma \circ \rho$ from D onto itself l-p-nulls X. Let $f : X \to D$ be a map. Since $\rho \circ f$ is a map from X into Y, $\gamma \circ (\rho \circ f)$ is null-pseudo-homotopic. This implies that $\mu \circ \gamma \circ \rho \circ f$ is null-pseudo-homotopic. □

Corollary 4.11. *Let X and Y be continua. If X is left g-pseudo-contractible (left g-contractible) with respect to Y and X and Y are continuously equivalent, then X and Y are g-pseudo-contractible (g-contractible).*

Let X be a topological space and let A be a closed subset of X. A map $r : X \to A$ is a *retraction* provided that $r|_A$ is the identity map on A. The set A is called a retract of X. We say that A is a deformation retract of X or A is a pseudo-deformation retract of X if there exists a retraction $r : X \to A$ which is homotopic to the identity map on X or is pseudo-homotopic to the identity map on X, respectively.

Theorem 4.12. *Let X, Y be continua and let A be a retract of X. If X is left g-pseudo-contractible (left g-contractible) with respect to Y, then A is left g-pseudo-contractible (left g-contractible) with respect to Y.*

Proof. Let $\gamma : Y \to Y$ be an onto map that l-p-nulls X. We shall prove that γ l-p-nulls A too. Let $h : A \to Y$ be a map. Let $r : X \to A$ be a retraction, let C be a continuum and let $H : X \times C \to Y$ be a pseudo-homotopy between $\gamma \circ (h \circ r)$ and a constant map $l : X \to Y$. Then, the restriction map $H|_{A\times C}$ is a pseudo-homotopy between the maps $\gamma \circ h$ and $l|_A$. □

Let X and Y be topological spaces such that $X \cap Y = \emptyset$. Given $w \in X$ and $u \in Y$, the quotient space obtained by the partition

$$\{\{w,u\}\}\cup\{\{x\}:x\in X\cup Y\setminus\{w,u\}\}$$

of the free union of X and Y (see [13, Definition 3.17, p. 42] for the definition of the free union of two spaces) will be denoted by $(X,u)\vee(Y,w)$ and it is called the adjunction space.

It is well known that if X and Y are disjoint continua, then the adjunction space $(X,w)\vee(Y,u)$ is a continuum (see [13, Theorem 3.20, p. 43]).

Theorem 4.13. *Let X, Y be continua and let A be a pseudo-deformation retract of X. Then, A is left g-pseudo-contractible with respect to Y if and only if X is left g-pseudo-contractible with respect to Y.*

Proof. Suppose that A is left g-pseudo-contractible with respect to Y, there exists an onto map $\gamma:Y\to Y$ such that γ l-p-nulls A. In order to prove that γ also l-p-nulls X, we consider a map $f:X\to Y$. Since γ l-p-nulls A, there exist a continuum C, points $u,v\in C$ and a map $G:A\times C\to Y$ such that $G(a,u)=\gamma\circ f\mid_A(a)$ and $G(a,v)=l$ for some $l\in Y$. On the other hand, from our assumption on A, it follows that there exist a retraction $r:X\to A$, a continuum D, points $z,w\in D$ and a map $F:X\times D\to X$ satisfying $F(x,z)=x$ and $F(x,w)=r(x)$.

We may assume that $C\cap D=\emptyset$. We define $H:X\times((C,u)\vee(D,w))\to Y$ by

$$H(x,t)=\begin{cases}\gamma(f(F(x,t))), & \text{if } t\in D,\\[2ex] G(r(x),t), & \text{if } t\in C,\end{cases}$$

to get a pseudo-homotopy between $\gamma\circ f$ and l. Therefore, γ l-p-nulls X.

On the other hand, by Theorem 4.12 the part “only if” holds. □

Theorem 4.14. *Let X, Y be continua and let B be a retract of Y. If X is right g-pseudo-contractible (right g-contractible) with respect to Y, then X is right g-pseudo-contractible (right g-contractible) with respect to B.*

Proof. Let us prove that each map from X onto itself which r-p-nulls Y, r-p-nulls B too. Let $\alpha:X\to X$ be an onto map such that r-p-nulls Y and let $h:X\to B$ be a map. Now, since $h\circ\alpha$ is a map from X to Y, there exists a pseudo-homotopy H between $h\circ\alpha$ and a constant map l. By hypothesis there exists a retraction $r:Y\to B$. Then, the map $r\circ H$ is a pseudo-homotopy between $h\circ\alpha$ and the constant map $r\circ l$. Thus, α r-p-nulls B. □

Theorem 4.15. *Let X, Y be continua and let B be a pseudo-deformation retract of Y. Then, X is right g-pseudo-contractible with respect to B if and only if X is right g-pseudo-contractible with respect to Y.*

Proof. The only if part is given by Theorem 4.14. To prove the if part, let $\alpha:X\to X$ be an onto map such that α r-p-nulls B. We will prove that α r-p-nulls Y too. Let f be a map from X into Y. By hypothesis, there exist a retraction $r:Y\to B$, a continuum D, points $z,w\in D$ and a map $G:Y\times D\to B$ such that $G(y,z)=y$ and $G(y,w)=r(y)$ for each $y\in Y$. On the other hand, there exist a continuum C, points $u,v\in C$ and a map $F:X\times C\to B$ such that $F(x,u)=r(f(\alpha(x)))$ and $F(x,v)=l$ for each $x\in X$ and for some $l\in B$.

We may assume that $C\cap D=\emptyset$. We define $H:X\times((C,u)\vee(D,w))\to Y$ a pseudo-homotopy between $f\circ\alpha$ and l given by

$$H(x,t)=\begin{cases}G(f(\alpha(x)),t), & \text{if } t\in D,\\[2ex] F(x,t), & \text{if } t\in C.\end{cases}$$

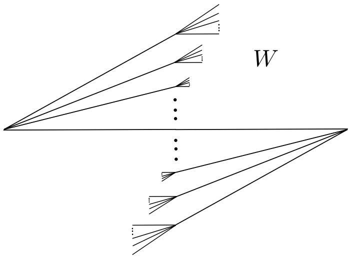

Fig. 3. Continuum W of Example 5.1.

In conclusion, α r-p-nulls Y. $\square$

5. Maps and g-pseudo-contractibility

Given a topological property $\mathcal{P}$ and a class $\mathcal{M}$ of maps, we say that $\mathcal{P}$ is *invariant* under the class $\mathcal{M}$ provided each onto map $f : X \to Y$ with $f \in \mathcal{M}$ if X has $\mathcal{P}$ then Y has $\mathcal{P}$.

Let X, Y be continua. An onto map $f : X \to Y$ is said to be:

- *Open*, provided that the image of each open subset of X under f is an open subset of Y.
- *Monotone* if the pre-image of each point of Y is connected.
- *Light*, if for each point of Y, every component of its pre-image is degenerate.

This section is dedicated to show that the property of being g-pseudo-contractible is not invariant under retractions, open maps and the property of not being g-pseudo-contractible is not invariant under retractions, open maps and monotone maps. We will use the following notation and definitions. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of subsets of a space X. We define the limit inferior of $\{A_n\}_{n\in\mathbb{N}}$, denoted by $\liminf A_n$ as the set of all $x \in X$ such that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to x with $x_n \in A_n$ for every $n \in \mathbb{N}$. Let X be a continuum and let Y be a subset of X. We denote by $NLC(Y)$ the set of all points of Y at which X is not locally connected. Also, $\mathrm{diam}(Y)$ will denote the diameter of the subset Y. Finally, the set of all components of Y in X will be denoted by $comp(Y)$. If $u, w \in \mathbb{R}^2$, we denote by uw the straight line segment with endpoints u and w. We define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ as the map obtained by the reflection through the origin.

The continuum W below appears in [3], here it is described for our purposes.

Example 5.1. Continuum W: Let $a = (-1, 0)$, $p = (0, 0)$ and $d_n = (0, \frac{1}{n})$ for each $n \in \mathbb{N}$. Let F_H be the Harmonic Fan defined by:

$$F_H = \left(\bigcup_{n\in\mathbb{N}} ad_n \right) \cup ap.$$

Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of continua which are homeomorphic to F_H such that $\{diamD_n\}_{n\in\mathbb{N}}$ converges to 0. We define the continuum X_H by:

$$X_H = F_H \cup \left(\bigcup_{n\in\mathbb{N}} D_n \right).$$

Where we identify the vertex of D_n with d_n.

The continuum W is defined by $W = X_H \cup \phi(X_H)$ (See Fig. 3.)

Proposition 5.2. *The continuum W described in Definition 5.1 is not g-pseudo-contractible.*

Proof. Suppose that W is g-pseudo-contractible. Then, there exist a continuum C, points $c, d \in C$, an onto map $g : W \to W$ and a pseudo-homotopy $H : W \times C \to W$ satisfying $H(x, c) = g(x)$ and $H(x, d) = l$ for some $l \in W$.

Let $U = W \setminus \{a, \phi(a)\}$. For each $n \in \mathbb{N}$, let $C_{2n-1} = D_n \cup (ad_n \setminus \{a\})$ and $C_{2n} = \phi(D_n) \cup (\phi(a)\phi(d_n) \setminus \{\phi(a)\})$. Thus, U is an open subset of W, $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of components of U and $\liminf C_n = \{p\}$.

Claim 1. There exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ in W such that $\lim p_n = p$ and $p_n \in g^{-1}(C_n)$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let $L_{2n-1} = NLC(D_n)$, $L_{2n} = NLC(\phi(D_n))$ and $L_0 = NLC(a\phi(a))$. Observe that $NLC(W) = \bigcup\{L_n : n \in \mathbb{N} \cup \{0\}\}$. Now, we take $w_n \in L_n$ for each $n \in \mathbb{N}$. Then, $\{w_n\}_{n\in\mathbb{N}}$ is a sequence such that $\lim w_n = p$. By [3, Lemma 4.11, p. 471], $NLC(W) \subseteq g(NLC(W))$. Then, there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ in $NLC(W)$ such that $g(p_n) = w_n \in L_n \subseteq C_n$. We will prove that $\lim p_n = p$.

Assume that there exists $m \in \mathbb{N} \cup \{0\}$ such that $J = \{n \in \mathbb{N} : p_n \in L_m\}$ is an infinite set. The fact that $\overline{L_m}$ is an arc contained in W implies that $g(\overline{L_m})$ is a local connected subcontinuum of W such that $\{w_n : n \in J\} \subseteq g(\overline{L_m})$, a contradiction. Thus, if $m \in \mathbb{N} \cup \{0\}$, then $J_m = \{n \in \mathbb{N} : p_n \in L_m\}$ is either a finite set or an empty set. Let $\{m_k\}_{k=1}^{\infty}$ be the increasing sequence consisting of all elements of $\{m \in \mathbb{N} : J_m \neq \emptyset\}$. Now, let $\rho > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that $\overline{L_{m_k}} \subseteq (-\rho, \rho) \times (-\rho, \rho)$ for each $k \geq k_0$. Finally, let $N = \max \bigcup\{J_{m_k} : k < k_0\}$. We shall prove that $p_n \in (-\rho, \rho) \times (-\rho, \rho)$ for each $n \geq N$. Observe that if $j < k_0$ and $n \in \mathbb{N}$ is such that $p_n \in L_{m_j}$, then $n \in J_{m_j} \subseteq \bigcup\{J_{m_k} : k < k_0\}$ and so $n < N$. Hence, we conclude that if $n \geq N$ and $j \in \mathbb{N}$ are such that $p_n \in L_{m_j}$, then $j \geq k_0$ and this implies that $p_n \in \overline{L_{m_j}} \subseteq (-\rho, \rho) \times (-\rho, \rho)$. Therefore, $\lim p_n = p$. The proof of Claim 1 is finished.

Let $\varepsilon > 0$ such that $d(p, W \setminus U) = \min\{d(p, x) : x \in W \setminus U\} \geq \varepsilon$. From the fact that H is uniformly continuous map, it follows there exists $\delta > 0$ such that $d_W(H((x, s)), H(y, r))) < \varepsilon$ provided $d_{W\times C}((x, s), (y, r)) < \delta$. Set $P = \{p\} \cup \{p_n : n \in \mathbb{N}\}$. Suppose that $\operatorname{diam}(P) < \delta$. Then, $\operatorname{diam}(H(P \times \{z\})) < \varepsilon$ for each $z \in C$. Set $V = \{t \in C : H(P \times \{t\}) \subseteq U\}$.

Claim 2. The set V is a non-empty open proper subset of C.

First, notice that $g(p) = p$. Thus $H(p, c) = p$ and $H(p_n, c) = g(p_n) \in C_n \subseteq U$. Therefore $c \in V$ and $V \neq \emptyset$.

In order to prove that $V \neq C$, we assume to the contrary that $V = C$. Then $\{l\} = H(P \times \{d\}) \subset U$. Let $M \in comp(U)$ such that $l \in M$. Let $j \in \mathbb{N}$. Notice that $H(\{p_j\} \times C)$ is a connected subset of U such that $l \in H(\{p_j\} \times C)$ and $H(\{p_j\} \times C) \subseteq C_j$. Therefore, $M = C_j$. Hence, $\overline{M} = \liminf C_n = \{p\} \subset U$. Thus, M is a component of the open subset U of W whose closure is contained in U. This condition contradicts the Boundary Bumping Theorem ([13, Theorem 5.7, p. 75]). Hence, $V \neq C$.

Now, we shall prove that V is an open subset of C. Let $t \in V$. Then $H(P \times \{t\}) \subseteq U$. Thus $P \times \{t\} \subseteq H^{-1}(U)$. The continuity of H and the Tube Lemma ([12, Lemma 26.8, p. 168]) ensure that there exist open subsets U_1 and U_2 of W and C, respectively, such that $P \times \{t\} \subseteq U_1 \times U_2 \subseteq H^{-1}(U)$. Since $P \times U_2 \subseteq U_1 \times U_2 \subseteq H^{-1}(U)$, we obtain that $t \in U_2 \subseteq V$. This concludes the proof of Claim 2.

Let V_0 be the component of V such that $c \in V_0$. Since $H(\{p_n\} \times V_0)$ is a connected subset of U and $H(p_n, c) = g(p_n) \in C_n$, we have that $H(\{p_n\} \times V_0) \subseteq C_n$ for each $n \in \mathbb{N}$. Now, observe that if $m \in \overline{V_0}$, then there exists a sequence $\{m_n\}_{n\in\mathbb{N}}$ in V_0 such that it converges to m and so $\{H(p_n, m_n)\}_{n\in\mathbb{N}}$ is a sequence in W converging to $H(p, m)$ and $H(p_n, m_n) \in H(\{p_n\} \times V_0)$ for each $n \in \mathbb{N}$. Thus, $H(\{p\} \times \overline{V_0}) \subseteq \liminf H(\{p_n\} \times V_0) \subseteq \liminf C_n = \{p\}$.

The fact that V_0 is a component of V and the Boundary Bumping Theorem imply that $\overline{V_0} \cap (C \setminus V) \neq \emptyset$. Let $t \in \overline{V_0} \cap (C \setminus V)$. Then, $t \in Bd(V_0) \setminus V$. Thus, $H(P \times \{t\})$ is not contained U, it follows that there exists $x \in P$ such that $H(x, t) \notin U$. Using the condition $H(p, t) = p$ and the choice of ε, we deduce that

$d(H(p,t), H(x,t)) \geq \varepsilon$. Since H is a uniformly continuous map and $\operatorname{diam}(P) < \delta$, thus $\operatorname{diam}(H(P \times \{t\})) < \varepsilon$. This is a contradiction. Therefore, the continuum W is not g-pseudo-contractible. □

Proposition 5.3. *The continuum W described in Example 5.1 fulfills each one of the following statements:*

1. *W is not g-pseudo-contractible.*
2. *$W \times I$ is g-pseudo-contractible.*
3. *There exists an open monotone retract from a g-pseudo-contractible continuum onto W.*
4. *There exists a monotone map from a contractible continuum onto W.*
5. *There exists a light open map from W onto a contractible continuum.*
6. *There exists a retraction from W onto a contractible continuum.*
7. *There exists a monotone map from W onto a contractible continuum.*

Proof. The statement (1) was proved in Proposition 5.2. Invoke [3, Theorem 4.19, p. 473] and Theorem 3.1 to show (2).

In order to prove (3), let $f : W \times I \to W$ be a map defined by $f(w,t) = w$ for each $(w,t) \in W \times I$. Notice that f is an open map and also f is a retraction from $W \times I$ onto $W \times \{0\}$. From (2), $W \times I$ is g-contractible.

Let us prove (4). Let Z be defined by $Z = X_H \cup p\phi(a) \cup (\phi(a) + \phi(X_H))$. Notice that the continuum Z is contractible. Let $\gamma : Z \to \gamma(Z)$ be the quotient map that result of collapsing the set $p\phi(a)$ to a single point. Observe that $\gamma(Z)$ is homeomorphic to W. Therefore $\gamma(Z)$ is not g-contractible. Notice that γ is a monotone map. This proves (4).

Now we will prove (5). We consider $\eta : W \to X_H$ the map defined by $\eta(w,y) = (w,y)$ if $(w,y) \in X_H$ and $\eta(w,y) = \phi(w,y)$ otherwise. The map η is a light open retraction and the continuum X_H is contractible.

Finally, let $\sigma : W \to X_H$ be a map defined by $\sigma(w,y) = (w,y)$ if $(w,y) \in X_H$ and $\sigma(w,y) = p$ otherwise. Notice that σ is a monotone retraction from W to X_H. This shows (6) and (7). □

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$G\mathcal{R}$-SETS AND G-PSEUDO-CONTRACTIBILITY.

José G. Anaya, Félix Capulín, Lucero Madrid-Mendoza, and David Maya

Abstract. A topological space X is said to be *g-pseudo-contractible* provided that there exist an onto map $f : X \to X$, a continuum C, points $a, b \in C$ and a map $H : X \times C \to X$ such that $H(x, a) = f(x)$ and $H(x, b) = p$ for some $p \in X$. The main purpose of this paper is to present conditions that obstruct g-contractibility and g-pseudo-contractibility with respect to. In this sense, we introduce the notions $g\mathcal{R}$-set and $g\mathcal{R}$-set with respect to and we present properties related to these concepts.

1. Introduction

A *continuum* is a non-empty compact connected metric space. A continuum is *contractible* provided that the identity map on itself is homotopic to a constant map. A general problem in Continuum Theory is to give conditions that imply the non-contractibility of a continuum, one of them is the existence of subsets that obstruct it. S. T. Czuba introduced in [7] the concepts of R^i-continua $(i = 1, 2, 3)$ in the class of dendroids in order to prove that each dendroid containing some one of them is not contractible. W. J. Charatonik in [6] extended Czuba's results to the class continua; additionally, he proved that if a continuum has an R^i continuum $(i = 1, 2, 3)$, then certain hyperspaces contain some R^i-continuum and as a consequence they are not contractible. Finally B. S. Baik, K. Hur and C. J. Rhee introduced in [2] the concept of R^i-sets. It is a natural generalization of R^i-continuum. Among other things, they gave a relationship among R^i-sets and they proved that the hyperspaces 2^X and $C(X)$ of continuum containing any one of the R^i-sets $(i = 1, 2, 3)$, also contain some R^i-set. As a consequence, these hyperpaces are not contractible.

On the other hand, a continuum is *g-pseudo-contractible* provided that there exists an onto map which is pseudo-homotopic to a constant map. The g-pseudo-contractibility is a natural generalization of notions: contractibility, pseudo-contractibility and g-contractibility and was introduced in [1]. One of the purposes of this paper is to wide the study of this concept was made in [1] presenting conditions that imply the non-g-pseudo-contractibility of a continuum which are related to R^i-sets.

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After Introduction and Preliminares, in Section 3, we introduce the concepts of a set g-pseudo-homotopically fixed and gR-set proving mainly that every continuum containing a g-pseudo-homotopically fixed set is not g-pseudo-contractible. In Section 4, we use the concepts of left (right) g-pseudo-contractible with respect to with the purpose of to introduce the notions of left (right) g-pseudo-homotopically fixed subset of a space with respect to other one and left (right) gR-set of a space with respect to other one and to prove some relationship between them.

2. Preliminaries

We denote by I the unit interval of the real line and $\mathbb{N}$ denotes the set of all positive integers. A *map* means a continuous function between topological spaces. Let X be a topological space and let A be a subset of X. We will use $\overline{A}$ and $\mathrm{Bd}(A)$ to represent its closure in X and its boundary in X, respectively. The set of all components of A in X will be represented by $\mathrm{Comp}(A)$.

The identity map on a space Z is denoted by id_Z. Let f be a map between topological spaces X and Y. If there exist a continuum C, points $a, b \in C$ and a map $H : X \times C \to Y$ such that $H(x, a) = f(x)$ and $H(x, b) = p$ for some $p \in Y$, then f is called *null-pseudo-homotopic*. For the other hand, the map f is said to be *null-homotopic* if there exists a map $H : X \times I \to Y$ satisfying $H(x, 0) = f(x)$ and $H(x, 1) = p$ for some $p \in Y$.

A continuum X is said to be:

- *Contractible* if the identity map on X is null-homotopic.
- *Pseudo-contractible* if the identity map on X null-pseudo-homotopic.
- *g-contractible* provided that there exists a an onto map $f : X \to X$ such that f is null-homotopic.
- *g-pseudo-contractible* provided that there exists a an onto map $f : X \to X$ such that f is null-pseudo-homotopic.

Some results related to these concepts can be consulted in [1], [3], [4], [5] and [8]. In [1], the following result is proved and suitable examples that show the converse of each one of the statements fails are presented.

Theorem 2.1. [1, Theorem 3.1, p. 2] *Let X be a continuum.*

(1) If X is contractible, then X is g-contractible.
(2) If X is contractible, then X is pseudo-contractible.
(3) If X is g-contractible, then X is g-pseudo-contractible.
(4) If X is pseudo-contractible, then X is g-pseudo-contractible.

3. gR-sets

Let X and C be continua. A map $H : X \times C \to X$ is called a *g-pseudo-deformation* if $H(X \times \{a\}) = X$ for some $a \in C$. If a map $H : X \times I \to X$ satisfies $H(X \times \{0\}) = X$, then H is called a *g-deformation*. Each map $H : X \times I \to X$ fulfilling $H(x, 0) = x$ for each $x \in X$ is called a *deformation*. Observe that each deformation H satisfies $H(X \times \{0\}) = X$. So, each deformation is a g-deformation. Since I is a continuum, if H is a g-deformation, then H is a g-pseudo-deformation.

A non-empty proper subset K of a continuum X is called:

- *g-pseudo-homotopically fixed* if $H(K \times C) = H(K \times \{a\}) \subsetneq X$ for each continuum C, for each g-pseudo-deformation $H : X \times C \to X$ and for each $a \in C$ such that $H(X \times \{a\}) = X$.
- *g-homotopically fixed* if $H(K \times I) = H(K \times \{0\}) \subsetneq X$ for each g-deformation $H : X \times I \to X$.
- *homotopically fixed* if $H(K \times I) = K$ for each deformation $H : X \times I \to X$.

The following result shows all possible relationships among these concepts.

Theorem 3.1. *Let X be a continuum and let K be a non-empty proper subset of X. Each one of the following statements holds.*

1. *If K is g-pseudo-homotopically fixed, then K is g-homotopically fixed.*
2. *If K is g-homotopically fixed, then K is homotopically fixed.*
3. *If K is g-pseudo-homotopically fixed, then K is homotopically fixed.*

Proof. We shall prove (1). Let $H : X \times I \to X$ be a g-deformation. Thus, H is a g-pseudo-deformation such that $H(X \times \{0\}) = X$. Since K is g-pseudo-homotopically fixed, we have $H(K \times I) = H(K \times \{0\}) \subsetneq X$.

In order to prove (2), let $H : X \times I \to X$ be a deformation. Then, H is a g-deformation. Now, from the fact that K is g-homotopically fixed, it follows that $H(K \times I) \subseteq H(K \times \{0\}) \subsetneq X$.

Finally, (1) and (2) together imply (3). □

In Example 3.6, we will show that the converse of the statements 2. and that of 3. of Theorem 3.1 are not true.

If a continuum contains a subset which is homotopically fixed, then it is not contractible (see [2, p. 3.15]). Thus, by Theorem 3.1, if a continuum contains a g-pseudo-homotopically fixed subset, then the continuum is not contractible and so, if a continuum contains a g-homotopically fixed subset, then the continuum is not contractible. The next result generalizes both facts.

Theorem 3.2. *Each continuum containing a g-homotopically fixed subset is not g-contractible.*

Proof. Let X be a continum and let K be a g-homotopically fixed subset of X. Suppose that X is g-contractible. Thus, there exists a map $H : X \times I \to X$ such that $H(X \times \{0\}) = X$ and, for each $x \in X$, $H(x, 1) = p$, for some $p \in X$. Thus, H is a g-deformation. Since K is g-homotopically fixed, we obtain that $H(K \times I) = H(K \times \{0\}) \subsetneq X$. Let $q \in X \setminus H(K \times \{0\})$.

Define $G : X \times I \to X$ by

$$G(x,t)=\begin{cases} H(x,2t), & \text{if } t\leq \frac{1}{2},\\ H(q,2(1-t)), & \text{if } \frac{1}{2}\leq t. \end{cases}$$

Notice that G is a g-deformation. Thus, $G(z,1)\in G(K\times I)=G(K\times\{0\})=H(K\times\{0\})$ for each $z\in X$. Therefore, $G(q,1)=H(q,0)\in H(K\times\{0\}$. A contradiction. □

Let X and Y be topological spaces such that $X\cap Y=\emptyset$. Given $w\in X$ and $u\in Y$, the quotient space obtained by the partition

$$\{\{w,u\}\}\cup\{\{x\}:x\in X\cup Y\setminus\{w,u\}\}$$

of the free union of X and Y (see [10, Definition 3.17, p. 42] for the definition of the free union of two spaces) is called the *adjuntion space* and it will be denoted by $(X,u)\vee(Y,w)$.

It is well known that if X and Y are disjoint continua, then the adjuntion space $(X,w)\vee(Y,u)$ is a continuum (see [10, Theorem 3.20, p. 43]).

Theorem 3.3. *Each continuum containing a g-pseudo-homotopically fixed subset is not g-pseudo-contractible.*

Proof. Let X be a continuum and let K be a g-pseudo-homotopically fixed subset of X. Suppose that X is g-pseudo-contractible. So, there exist a continuum C, points $a,b\in C$ and a map $H:X\times C\to X$ satisfying $H(X\times\{a\})=X$ and $H(X\times\{b\})=\{p\}$ for some $p\in X$. Thus, H is a g-pseudo-deformation. Let $q\in X\setminus H(K\times\{a\})$. We may assume that $X\cap C=\emptyset$. Define $G: X\times((C,b)\vee(X,p))\to X$ by

$$G(x,t)=\begin{cases} H(x,t), & \text{if } t\in C,\\ t, & \text{if } t\in X. \end{cases}$$

Notice that G is a map satisfying $G(X\times\{a\})=H(X\times\{a\})=X$. Then G is a g-pseudo-deformation such that $q=G(x,q)\in G(K\times((C,b)\vee(X,p)))$ and $q\notin G(K\times\{a\}))$. This contradicts the fact that $G(K\times((C,b)\vee(X,p))=G(K\times\{a\})$. □

Let X be a continuum and let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of non-empty subsets of a space X. The *limit inferior* of $\{A_n\}_{n\in\mathbb{N}}$, denoted by $\liminf A_n$, is the set of all $x\in X$ such that there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to x, with $x_n\in A_n$ for every $n\in\mathbb{N}$. The *limit superior* of $\{A_n\}_{n\in\mathbb{N}}$, denoted by $\limsup A_n$, is the set of all $x\in X$ such that there exist an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim x_{n_k}=x$ and $x_{n_k}\in A_{n_k}$ for each $k\in\mathbb{N}$.

A non-empty closed proper subset K of a continuum X is called an *R^3-set* if there exist an open subset U of X containing K and a sequence $\{C_n\}_{n\in\mathbb{N}}\in Comp(U)$ such that $K=\liminf C_n$. By [2, Theorem 2.3 and Corollary 2.5, p.

310], if a continuum contains an R^i-set (see [2, Definition 2.2, p. 310]), then such continuum contains a R^3-set too.

The following result are related with the above definitions.

Theorem 3.4. [2, Theorem 3.2, p. 315] *Let X be a continuum and let K be a proper subset of X. If K is an R^3-set, then K is homotopically fixed.*

Theorem 3.5. [5, Theorem 4.3, p. 366] *If a continuum X contains an R^3-set, then X is not pseudo-contractible.*

The continuum described below shows that the converse of 2. and that of 3. in Theorem 3.1 are not true. We will use the following notation. If $u, w \in \mathbb{R}^2$, we denote by uw the straight line segment with end points u and w. We define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ as the map obtained by the reflection through the origin.

Two continua X and Y are said to be *continuosly equivalent* if there exist two onto maps $\mu : X \to Y$ and $\phi : Y \to X$.

Example 3.6. Let $a = (-1, 0)$, $p = (0, 0)$ and $d_n = (0, \frac{1}{n})$ for each $n \in \mathbb{N}$. Let F_H be the harmonic fan defined by:

$$F_H = (\bigcup_{n\in\mathbb{N}} ad_n) \cup ap$$

Let Y_H be the continuum defined by:

$$Y_H = F_H \cup \phi(F_H).$$

Notice that $\{p\}$ is an R^3-set of Y_H. Therefore, by Theorem 3.4, $\{p\}$ is homotopically fixed.

Since Y_H is continuously equivalent to F_H and F_H is g-contractible, then by [3, Remark 2.15, p. 464]), Y_H is g-contractible. By Theorem 3.2, $\{p\}$ is not a g-homotopically fixed subset of Y_H. Furthermore, applying [1, Theorem 3.1, p. 2], we can conclude that Y_H is g-pseudo-contractible. Invoke Theorem 3.3 to conclude that $\{p\}$ is not a g-pseudo-homotopically fixed subset of Y_H.

A non-empty proper closed subset K of a continuum X is called $gR - set$ of X if for each onto map $f : X \to X$ there exist a proper open subset U of X and a sequence $\{C_n\}_{n\in\mathbb{N}}$ in Comp (U) such that:

(1) $K \subseteq f^{-1}(U)$;
(2) $K \subseteq \liminf f^{-1}(C_n)$; and
(3) $\liminf C_n \subseteq f(K)$.

Observe that if K is a gR-set of a continuum X, then K is a R^3-set of a continuum X. In the following example we will show the existence of gR-sets. The example appears in [3]. Here we repeat the construction for our purposes.

Example 3.7. Let F_H and Y_H be the continua described in Example 3.6. Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of fans pairwise disjoint, which are homeomorphic to

F_H and the sequence of diameter $\{\mathrm{diam}(D_n)\}_{n\in\mathbb{N}}$ converges to 0. We define the continuum X_H by:

$$X_H = F_H \cup (\bigcup_{n\in\mathbb{N}} D_n),$$

where, we identify the vertex of D_n with d_n for each $n \in \mathbb{N}$.

The continuum W is defined by

$$W = X_H \cup \phi(X_H)$$

(See Figure 1). Now, we denote $E_n = \phi(D_n)$ and $e_n = \phi(d_n)$ for each $n \in \mathbb{N}$.

We shall prove that $\{p\}$ is a gR-set. Let $f : W \to W$ be an onto map. Set $U = W \setminus \{a, \phi(a)\}$. Then U is an open subset of W. For each $n \in \mathbb{N}$, let $C_{2n-1} = D_n \cup (ad_n \setminus \{a\})$ and $C_{2n} = E_n \cup (\phi(a)e_n \setminus \{\phi(a)\})$. Thus, $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of components of U. By [3, Lemma 4.15, p. 472], $p \in \limsup f^{-1}(e_n) \cap \limsup f^{-1}(d_n)$. Therefore, $f(p) = p \in U$. It follows that $p \in f^{-1}(U)$. Notice that $\liminf C_n = \{p\}$. So, 1) and 3) hold. Finally, we claim that $p \in \liminf f^{-1}(C_n)$. We define $L_{2n-1} = NLC(D_n)$, $L_{2n} = NLC(E_n)$ and $L_0 = NLC(a\phi(a))$. Now, take $w_n \in L_n$ for each $n \in \mathbb{N}$. Then $\{w_n\}_{n\in\mathbb{N}}$ is a sequence such that $\lim w_n = p$. Since $\bigcup_{n\in\mathbb{N}} L_n \subseteq f(\bigcup_{n\in\mathbb{N}} L_n)$ (see [8, Lemma 2, p. 2180]), there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ in $\bigcup_{n\in\mathbb{N}} L_n$ such that $f(z_n) = w_n \in L_n \subseteq C_n$. We will prove that $\lim z_n = p$. Suppose that there exists $m \in \mathbb{N} \cup \{0\}$ such that $J = \{n \in \mathbb{N} : z_n \in L_m\}$ is an infinite set. The fact that $\overline{L_m}$ is a locally connected subcontinuum of W implies that $f(\overline{L_m})$ is a locally connected subcontinuum of W satisfying that $\{w_n : n \in J\} \subseteq f(\overline{L_m})$, a contradiction. Thus, if $m \in \mathbb{N} \cup \{0\}$, then either $J_m = \{n \in \mathbb{N} : z_n \in L_m\}$ is a finite set is an empty set. Let $\{m_k\}_{k=1}^{\infty}$ be the increassing sequence consisting of all elements of $\{m \in \mathbb{N} : J_m \neq \emptyset\}$. Now, let V be an open subset in W such that $p \in V$. Since $\lim \overline{L_{m_k}} = \{p\}$, there exists $k_0 \in \mathbb{N}$ such that $\overline{L_{m_k}} \subseteq V$ for each $k \geq k_0$. Finally, let $N = \max \bigcup_{k<k_0} J_{m_k}$. Take $n > N$. Then $z_n \in \bigcup_{k\in\mathbb{N}} L_{m_k} \setminus \bigcup_{k<k_0} L_{m_k}$. Thus, there exists $j \in \mathbb{N}$ such that $j > k_0$ and $z_n \in L_{m_{k_j}} \subseteq \overline{L_{m_{k_j}}} \subseteq V$. Hence, $\lim z_n = p$ and $\{p\} \subseteq \liminf f^{-1}(C_n)$. The proof is completed.

Theorem 3.8. *Each gR-set in a continuum is g-pseudo-homotopically fixed.*

Proof. Let X be a continuum such that X contains a gR-set K. Suppose that there exist a continuum C, a point $a \in C$ and a map $H : X \times C \to X$ such that such that $H(X \times \{a\}) = X$ and $H(K \times C)$ is not contained in $H(K \times \{a\})$. Then, there exists $(z, s) \in K \times C$ such that $H(z, s) \notin H(K \times \{0\})$. Define the map $f : X \to X$ by $f(x) = H(x, a)$. Notice that f is an onto map. Since K is a gR-set, there exist a proper open subset U in X and a sequence $\{C_n\}_{n\in\mathbb{N}}$ in $\mathrm{Comp}(U)$ such that $K \subseteq f^{-1}(U)$, $K \subseteq \liminf f^{-1}(C_n)$ and $\liminf C_n \subseteq f(K)$.

Let S be an open subset of X such that $H(K \times \{a\}) \subseteq S \subseteq \overline{S} \subseteq U$ and $H(z, s) \notin \overline{S}$.

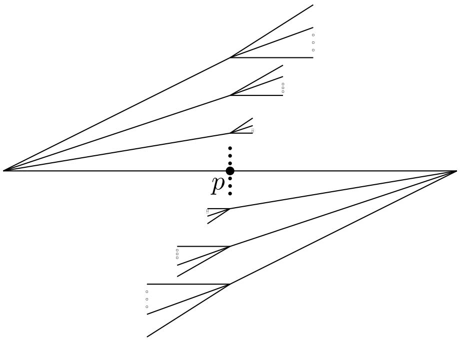

FIGURE 1. Continuum W.

Let $A = \{c \in C : H(K \times \{c\}) \subseteq S)\}$. Observe that $a \in A$ and $s \in C \setminus A$. Now, we shall prove that A is an open subset of C. Let $u \in A$. Then $H(K \times \{u\}) \subseteq S$. From the continuity of H and the Tube Lemma [9, Lemma 26.8, p. 168], it follows that there exist open subsets V_1 and V_2 of X and of C, respectively, such that $K \times \{u\} \subseteq V_1 \times V_2 \subseteq H^{-1}(S)$. Thus, $u \in V_2 \subset A$. Therefore, A is a non-empty proper open subset of C.

Let M be the component in A such that $a \in M$. Then, $H(K \times M) \subseteq S$. This implies that $H(K \times \overline{M}) \subseteq \overline{S} \subseteq U$.

Let $y \in K$. Since $K \subseteq \liminf f^{-1}(C_n)$, there exists a sequence $\{y_n\}_{n \in \mathbb{N}}$ in X satisfying $f(y_n) \in C_n$ for each $n \in \mathbb{N}$ and $\lim y_n = y$. Therefore, $\lim H(\{y_n\} \times \overline{M}) = H(\{y\} \times \overline{M}) \subseteq \overline{S} \subseteq U$. We may assume that each $H(\{y_n\} \times \overline{M}) \subseteq U$. Since $f(y_n) \in C_n$ and $f(y_n) = H(y_n, a) \in H(\{y_n\} \times \overline{M}) \in U$, we have that $H(\{y_n\} \times \overline{M}) \subseteq C_n$. Thus, $H(\{y\} \times \overline{M}) = \liminf H(\{y_n\} \times \overline{M}) \subseteq \liminf C_n \subseteq f(K) = H(K \times \{a\}) \subseteq S$. This implies that $H(K \times \overline{M}) \subseteq S$. Therefore, $\overline{M} \subseteq A$, this contradicts the Boundary Bumping Theorem [10, Theorem 5.7, p. 75]. In conclusion, K is g-pseudo-homotopically fixed. □

Theorem 3.9. *Each gR-set in a continuum is g-homotopically fixed.*

Proof. Let X be a continuum such that X contains a gR-set K. We shall show that K is g-homotopically fixed. Seeking a contradiction, suppose that there exists a g-deformation $H : X \times I \to X$ such that $H(K \times I)$ is not contained in $H(K \times \{0\})$. Then, there exists $(z, s) \in K \times I$ satisfying that

$$H(z, s) \notin H(K \times \{0\}).$$

Define the map $f : X \to X$ by $f(x) = H(x, 0)$. Notice that f is an onto map. Since K is a gR-set, there exist a proper open subset U in X and a sequence $\{C_n\}_{n \in \mathbb{N}}$ in $\operatorname{Comp}(U)$ such that $K \subseteq f^{-1}(U)$, $K \subseteq \liminf f^{-1}(C_n)$ and $\liminf C_n \subseteq f(K)$.

Since X is a continuum and $H(K \times \{0\})$ is a closed subset of X, there exists an open subset S of X such that $H(K \times \{0\}) \subseteq S \subseteq \overline{S} \subseteq U$ and $H(z, s) \in X \setminus \overline{S}$.

Let $t_p = \sup\{t \in I : H(\{p\} \times [0,t]) \subseteq S\}$ for each point $p \in K$ and let $r = \inf\{t_p : p \in K\}$. Then, $H(K \times [0,r]) \subseteq \overline{S}$.

Since $H(z,s) \notin \overline{S}$, $t_z < s \leq 1$. So, $r < 1$. Now, there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ in K such that the sequence $\{t_{p_n}\}_{n\in\mathbb{N}}$ converges to r. By the compactness of K we may assume that $\{p_n\}$ converges to some $q \in K$.

Observe that if $p \in K$ and $t_p < 1$, then $H(p, t_p) \in \mathrm{Bd}(S)$. Now, since $\lim t_{p_n} = r$ and $r < 1$, we can suppose that each $t_{p_n} < 1$. Thus,

$$H(p_n, t_{p_n}) \in \mathrm{Bd}(S)$$

for each $n \in \mathbb{N}$. From the continuity of H and the fact that $\mathrm{Bd}(S)$ is a closed subset of X, it follows that $H(q,r) \in \mathrm{Bd}(S)$. Hence, $r > 0$ and $H(q,r) \notin H(K \times \{0\})$.

The inclusion $K \subseteq \liminf f^{-1}(C_n)$ ensures that there exists a sequence $\{q_n\}_{n\in\mathbb{N}}$ in X such that $q_n \in f^{-1}(C_n)$ for each $n \in \mathbb{N}$ and $\lim q_n = q$ (see [10, Theorem 4.11, p. 57]). From the continuity of H,

$$\lim(H(\{q_n\} \times [0,r]) = H(\{q\} \times [0,r]) \subseteq \overline{S} \subseteq U.$$

So, there exists $N \in \mathbb{N}$ such that $H(\{q_n\} \times [0,r]) \subseteq U$ for each $n \geq N$. Since $f(q_n) \in C_n$ and $f(q_n) \in H(\{q_n\} \times [0,r]) \subseteq U$, we have that $H(\{q_n\} \times [0,r]) \subseteq C_n$ for each $n \geq N$. Therefore,

$$H(q,r) \in H(\{q\} \times [0,r]) = \lim(H(\{q_n\} \times [0,r]) \subseteq \liminf C_n \subseteq H(K \times \{0\}).$$

This is a contradiction. We deduce that K is g-homotopically fixed. □

The following result is an immediate consequence to combine Theorems 3.3 and 3.8 and to combine Theorems 3.2 and 3.9, respectively.

Theorem 3.10. *If a continuum X contains a gR-set then:*

(1) X is not g-pseudo-contractible.
(2) X is not g-contractible.

By [1, Theorem 3.17, p. 5] and Theorem 3.10 we have the following result.

Theorem 3.11. *Let X and Y be continua. If X contains a gR-set and X and Y are continuosly equivalent, then Y is not g-pseudo-contractible.*

4. gR-sets with respect to

Let X, Y and Z be topological spaces and let $\alpha : X \to X$ and $\gamma : Y \to Y$ be onto maps. We say that:

- the map γ *l-p-nulls Z (l-nulls Z)* provided that for each map $f : Z \to Y$, $\gamma \circ f$ is null-pseudo-homotopic (null-homotopic); and
- the map α *r-p-nulls Z (r-nulls Z)* provided that for each map $f : X \to Z$, $f \circ \alpha$ is null-pseudo-homotopic (null-homotopic).

The space X is called *left g-pseudo-contractible with respect to Y (left g-contractible with respect to Y)* if there exists an onto map $\gamma : Y \to Y$ such that γ l-p-nulls X (l-nulls X). We say that X is *right g-pseudo-contractible with respect to Y (right g-contractible with respect to Y)* if there exists an onto map $\alpha : X \to X$ such that α r-p-nulls Y (r-nulls Y). In Example 4.19 and Example 4.20, we will show that the concept of right g-pseudo-contractible with respect to is not equivalent to the concept of left g-pseudo-contractible with respect to. The reader interested in general properties of these concepts is referred to [1].

Let X and Y be spaces. We say that X is *pseudo-contractible (contractible) with respect to* Y provided that every map $f : X \to Y$ is null-pseudo-homotopic (null-homotopic).

Theorem 4.1. *Let X and Y be continua.*

(1) If X is pseudo-contractible, then X is right (left) g-pseudo-contractible with respect to Y.
(2) If X is pseudo-contractible, then Y is right (left) g-pseudo-contractible with respect to X.

Proof. We shall prove (1). By [4, Theorem 29, p. 64], we have that X is pseudo-contractible with respect to Y. Thus, each map $f : X \to Y$ is null-pseudo-homotopic. This implies that the $f \circ id_X$ and $id_Y \circ f$ are null-pseudo-homotopic for each map $f : X \to Y$. We infer that id_X r-p-nulls Y and id_Y l-p-nulls X. Therefore, X is right g-pseudo-contractible with respect to Y and X is left g-pseudo-contractible with respect to Y.

In order to prove that id_Y r-p-nulls X and id_X l-p-nulls Y, let $g : Y \to X$ be a map. Applying [4, Theorem 29, p. 64], we obtain that Y is pseudo-contractible with respect to X. Thus, g is null-pseudo-homotopic. This implies that the $g \circ id_Y$ and $id_X \circ g$ are null-pseudo-homotopic. Then, id_Y r-p-nulls X and id_X l-p-nulls Y. In conclusion, Y is right g-pseudo-contractible with respect to X and Y is left g-pseudo-contractible with respect to X. This shows (2). □

Since each contractible continuum is pseudo-contractible (see Theorem 2.1), the following result is an immediate consequence of Theorem 4.1.

Corollary 4.2. *Let X and Y be continua.*

(1) If X is contractible, then X is right (left) g-pseudo-contractible with respect to Y.
(2) If X is contractible, then Y is right (left) g-pseudo-contractible with respect to X.

In the rest of this section, we will give necessary conditions to obstruct left (right) g-pseudo-contractible with respect to. To this end, we introduce the following definitions.

Let X and Y be continua. A proper subset K of X is *left g-p-homotopically fixed of X with respect to Y (right g-p-homotopically fixed of X with respect* *to Y)* if for each onto map $f : Y \to Y$ ($f : X \to X$) there exists a map $h : X \to Y$ such that for each continuum C and for each map $G : X \times C \to Y$ such that $G(x, a) = f(h(x))$ ($G(x, a) = h(f(x))$) for some $a \in C$, we have $G(K \times C) = G(K \times \{a\}) \subsetneq Y$. If for each onto map $f : Y \to Y$ ($f : X \to X$) there exists a map $h : X \to Y$ such that for each map $G : X \times I \to Y$ such that $G(x, 0) = f(h(x))$ ($G(x, 0) = h(f(x))$) the inclusions $G(K \times I) = G(K \times \{0\}) \subsetneq Y$ holds, then we say that K is *right g-homotopically fixed of X with respect to Y (left g-homotopically fixed of X with respect to Y).*

Theorem 4.3. *Let X and Y be continua. If there exists a left g-p-homotopically fixed subset K of X with respect to Y, then X is not left g-pseudo-contractible with respect to Y.*

Proof. Suppose to the contrary that X is left g-pseudo-contractible with respect to Y. Thus, there exists an onto map $\gamma : Y \to Y$ such that γ l-p-nulls X. Now, since K is a subset left g-p-homotopically fixed of X with respect to Y, there exists a map $h : X \to Y$ such that for each continuum D and for each map $T : X \times D \to Y$ such that $T(x, u) = f(h(x))$ for some $u \in D$, we have $T(K \times D) = T(K \times \{u\}) \subsetneq Y$. From the fact that γ l-p-nulls X, there exist a continuum C, points $a, b \in C$ and a map $G : X \times C \to Y$ such that $G(x, a) = \gamma \circ h(x)$, $G(x, b) = q$ for each $x \in X$ and for some $q \in Y$. Observe that $\gamma \circ h(K) = G(K \times \{a\}) \subsetneq Y$. Thus, there exists $p \in Y \setminus \gamma \circ h(K)$. We may assume that $C \cap Y = \emptyset$. We define $H : X \times ((C, b) \vee (Y, q)) \to Y$ by

$$H(x,t) = \begin{cases} G(x,t), & \text{if } t \in C, \\ \\ t, & \text{if } t \in Y. \end{cases}$$

Thus, H is a map such that $H(x, a) = G(x, a) = \gamma \circ h(x)$. The assumption on h implies that $H(K \times ((C, b) \vee (Y, q))) = H(K \times \{a\}) \subsetneq Y$. Hence, if $z \in K$, then $p = H(z, p) \in H(K \times ((C, b) \vee (Y, q))) = H(K \times \{a\}) = \gamma \circ h(K)$, a contradiction. □

The proof of the following result is essentially the same that of Theorem 4.3, we include it for the sake of completeness.

Theorem 4.4. *Let X be a continuum, let K be a subset of X and let Y be an arcwise-connected continuum. If K is right g-homotopically fixed of X with respect to Y, then X is not right g-contractible with respect to Y.*

Proof. Suppose that X is right g-contractible with respect to Y. Let $\alpha : X \to X$ be an onto map such that α r-p-nulls Y. Since K is right g-homotopically fixed of X with respect to Y and α is an onto map, there exists a map $h : X \to Y$ satisfying that if $T : X \times I \to Y$ is a map such that $T(x, 0) = h \circ \alpha(x)$, then $T(K \times I) = T(K \times \{0\}) \subsetneq Y$. From the fact that α r-p-nulls Y, it follows that $h \circ \alpha$ is null-homotopic. Thus, there exists a map $G : X \times I \to Y$ such that $G(x, 0) = h \circ \alpha(x)$ and $G(x, 1) = q$ for some $q \in Y$. By the assumption on h, we have $G(K \times I) = G(K \times \{0\}) \subsetneq Y$. Let $p \in Y \setminus G(K \times \{0\})$. Let $\sigma : I \to Y$ be a map such that $\sigma(0) = q$ and $\sigma(1) = p$. Define $H : X \times I \to Y$ by

$$H(x,t) = \begin{cases} G(x, 2t), & \text{if } t \leq \frac{1}{2}, \\ \\ \sigma(2t-1), & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Note that H is a map such that $H(x, 0) = G(x, 0) = h \circ \alpha(x)$ and $H(x, 1) = p$. Then, $H(K \times I) = H(K \times \{0\}) \subsetneq Y$. Since $p \in H(x, 1)$, we obtain $p \in H(K \times \{0\}) = G(K \times \{0\})$, a contradiction. $\square$

The proof of the following two results is similar to Theorem 4.3 and Theorem 4.4, respectively.

Theorem 4.5. *Let X and Y be continua. If there exists a right g-p-homotopically fixed subset K of X with respect to Y, then X is not right g-pseudo-contractible with respect to Y.*

Theorem 4.6. *Let X be a continuum, let K be a subset of X and let Y be an arcwise-connected continuum. If K is left g-homotopically fixed of X with respect to Y, then X is not left g-contractible with respect to Y.*

Definition. Let X and Y be continua and let K be a proper closed subset of X. We say that K is a *right gR-set of X with respect to Y* if for each onto map $f : X \to X$ there exist a map $h : X \to Y$, a non-empty open proper subset U of Y and a sequence $\{C_n\}_{n\in\mathbb{N}}$ in $\text{Comp}(U)$ satisfying:

(r.1) $K \subseteq (h \circ f)^{-1}(U)$.
(r.2) $K \subseteq \liminf (h \circ f)^{-1}(C_n)$.
(r.3) $\liminf C_n \subseteq (h \circ f)(K)$

We say that K is a *left gR-set of X with respect to Y* if for each onto map $f : Y \to Y$ there exist a map $h : X \to Y$, a non-empty open proper subset U of Y and a sequence $\{C_n\}_{n\in\mathbb{N}}$ in $\text{Comp}(U)$ satisfying:

(l.1) $K \subseteq (f \circ h)^{-1}(U)$.
(l.2) $K \subseteq \liminf (f \circ h)^{-1}(C_n)$.
(l.3) $\liminf C_n \subseteq (f \circ h)(K)$.

Theorem 4.7. *Let X and Y be continua and let K be a subset of X. If K is a left gR-set of X with respect to Y, then K is left g-p-homotopically fixed of X with respect to Y.*

Proof. Let $f : Y \to Y$ be an onto map. There exist a map $h : X \to Y$, a proper open subset U of Y and a sequence $\{C_n\}_{n\in\mathbb{N}}$ in $\text{Comp}(U)$ satisfying (l.1), (l.2) and (l.3).

Now, let C be a continuum, $a \in C$ and let $G : X \times C \to Y$ be a map such that $G(x, a) = f(h(x))$. By (l.1), $G(K \times \{a\}) = f(h(K)) \subseteq U \subsetneq Y$. Seeking a contradiction, assume that $G(K \times C) \setminus G(K \times \{a\}) \neq \emptyset$. Let $(z, s) \in K \times C$ such that $G(z, s) \notin G(K \times \{a\})$.

Let V be a non-empty open subset of Y such that $f(h(K)) \subseteq V \subseteq \overline{V} \subseteq U$ and $G(z,s) \notin \overline{V}$. We define $A = \{c \in C : G(K \times \{c\}) \subset V\}$. Observe that $a \in A$ and $A \subseteq C \setminus \{s\}$. Now, we shall prove that A is an open subset of C. Let $y \in A$. Then $G(K \times \{y\}) \subseteq V$. Thus, $K \times \{y\} \subseteq G^{-1}(V)$. The continuity of G and the Tube Lemma [9, Lemma 26.8, p. 168] together imply that there exist open subsets V_1 and V_2 of X and C, respectively, such that $K \times \{y\} \subseteq V_1 \times V_2 \subseteq G^{-1}(V)$. Then, $y \in V_2 \subseteq A$. In conclusion, A is an non-empty proper open subset of C.

Let M be the component of A such that $a \in M$. Thus, $G(K \times M) \subseteq V$. This implies $G(K \times \overline{M}) \subseteq \overline{V}$.

Let $w \in K$. By (1.2) there exists a sequence $\{w_n\}_{\in \mathbb{N}}$ in X such that $f(h(w_n)) \in C_n$ for each $n \in \mathbb{N}$ and $\lim w_n = w$. Thus, $\lim G(\{w_n\} \times \overline{M}) = G(\{w\} \times \overline{M}) \subseteq \overline{V} \subseteq U$. Hence we may assume that each $G(\{w_n\} \times \overline{M}) \subseteq U$. Since $f(h(w_n)) = G(w_n, a) \in G(\{w_n\} \times \overline{M}) \subseteq U$, $G(\{w_n\} \times \overline{M})$ is connected and $f(h(w_n)) \in C_n$ for each $n \in \mathbb{N}$, each $G(\{w_n\} \times \overline{M}) \subseteq C_n$. Applying (1.3) we obtain $G(\{w\} \times \overline{M}) = \lim G(\{w_n\} \times \overline{M}) \subseteq \liminf C_n \subseteq f(h(K)) = G(K \times \{a\}) \subseteq V$. Therefore, $G(K \times \overline{M}) \subseteq V$. Then, $\overline{M} \subseteq A$. This condition contradicts the Boundary Bumping Theorem [10, Theorem 5.7, p. 75]. In conclusion $G(K \times C) \subseteq G(K \times \{a\})$. ☐

Similar arguments in the proof of last result prove the next one.

Theorem 4.8. *Let X and Y be continua and let K be a subset of X. If K is a right gR-set of X with respect to Y, then K is right g-p-homotopically fixed of X with respect to Y.*

The following result is an immediate consequence of to apply Theorems 4.3, 4.5, 4.7 and 4.8.

Theorem 4.9. *Let X and Y be continua and let K be a subset of X. If K is left (right) gR-set of X with respect to Y, then X is not left (right) g-pseudo-contractible with respect to Y.*

Theorem 4.10. *Let X, D and Y be continua. If X contains a right gR-set with respect to Y and X is continuosly equivalent to D, then D contains a right gR-set with respect to Y.*

Proof. Let K be a right gR-set of X with respect to Y. Since X and D are continuosly equivalent, there exist onto maps $\mu : X \to D$ and $\phi : D \to X$. We shall prove that $\mu(K)$ is a right gR-set of D with respect to Y.

Let $f : D \to D$ be an onto map. Then, $\phi \circ f \circ \mu : X \to X$ is an onto map. So, there exist a map $h : X \to Y$, a non-empty proper open subset U of Y, and a sequence $\{C_n\}_{n\in\mathbb{N}}$ in $\mathrm{Comp}(U)$ such that $K \subseteq (h \circ (\phi \circ f \circ \mu))^{-1}(U)$, $K \subseteq \liminf (h \circ (\phi \circ f \circ \mu))^{-1}(C_n)$ and $\liminf C_n \subseteq (h \circ (\phi \circ f \circ \mu))(K)$. Define $t = h \circ \phi$ to obtain a map from D to Y such that $\mu(K) \subseteq (t \circ f)^{-1}(U)$, $K \subseteq \liminf (t \circ f \circ \mu)^{-1}(C_n)$ and $\liminf C_n \subseteq (t \circ f)(\mu(K))$. Finally, the condition $K \subseteq \liminf(t \circ f \circ \mu)^{-1}(C_n)$ and the continuity of μ together imply $\mu(K) \subseteq \liminf(t \circ f)^{-1}(C_n)$. This finishes the proof. □

The following result is a consequence of [1, Theorem 4.10, p. 9] and Theorem 4.9.

Theorem 4.11. *Let X, D and Y be continua. If X contains a left gR-set with respect to Y and Y is continuosly equivalent to D, then X is not left g-pseudo-contractible with respect to D.*

By Theorem 4.9 and Theorem 4.10 we have the following result.

Theorem 4.12. *Let X, D and Y be continua. If X contains a right gR-set with respect to Y and X is continuosly equivalent to D, then D is not right g-pseudo-contractible with respect to Y.*

The proof of the next two results follows from Theorem 4.9 and [1, Theorem 4.12 and Theorem 4.15, pp. 9, 10].

Theorem 4.13. *Let X and Y be continua and let A be a retract of X. If A contains a left gR-set with respect to Y, then X is not left g-pseudo-contractible with respect to Y.*

Theorem 4.14. *Let X and Y be continua and let A be a pseudo-deformation retract of X. If X contains a left gR-set with respect to Y, then A is not left g-pseudo-contractible with respect to Y.*

By applying Theorem 4.9 and [1, Theorem 4.14 and Theorem 4.15, p. 10] we prove the following two results.

Theorem 4.15. *Let X and Y be continua and let B be a retract of Y. If X contains a right gR-set with respect to B, then X is not right g-pseudo-contractible with respect to Y.*

Theorem 4.16. *Let X and Y be continua and let B be a pseudo-deformation retract of Y. If X contains a right gR-set with respect to Y, then X is not right g-pseudo-contractible with respect a B.*

The result below follows from to apply [1, Theorem 4.6, p. 8] and Theorems 4.3, 4.9 and 4.13.

Corollary 4.17. *Let X and Y be continua. Each one of the following conditions implies that Y is non-g-pseudo-contractible.*

1. *There exists a left g-p-homotopically fixed subset of X with respect to Y.*
2. *There exists a left gR-set of X with respect to Y.*
3. *There exists a retract A of X such that A contains a left gR-set with respect to Y.*

By [1, Theorem 4.6, p. 8] and Theorems 4.5, 4.9 and 4.15, we have the following result.

Corollary 4.18. *Let X and Y be continua. Each one of the following conditions implies that X is non-g-pseudo-contractible.*

1. *There exists a right g-p-homotopically fixed subset of X with respect to Y.*
2. *There exists a right gR-set of X with respect to Y.*
3. *There exists a retract B of Y such that X contains a right gR-set with respect to B.*

The following example shows the existence of left gR-set, the condition of being right g-pseudo-contractible with respect to a space does not imply the condition of being left g-pseudo-contractible with respect to the same space and that the converse of [1, Theorem 4.12, p. 9] is not true.

Example 4.19. Let W be the continuum defined in Example 3.7. There exists a continuum Y satisfying the following conditions:

1. The continuum Y is right g-pseudo-contractible with respect to W.
2. $\{p\}$ is a left gR-set of Y with respect to W.
3. The continuum Y is not left g-pseudo-contractible with respect to W.
4. There exists a retract A of Y such that A is left g-pseudo-contractible with respect to W.

First we will define the continuum Y. Consider the continuum F_H defined in Example 3.6. Now, let $d_n^{'}$ be the point $(\frac{1}{2n}, \frac{1}{n})$ for each $n \in \mathbb{N}$. We define the continuum

$$F = F_H \cup (\bigcup_{n \in \mathbb{N}} d_n d_n^{'}).$$

Finally the continuum Y is defined by $Y = F \cup \phi(F)$.

Note that the continuum Y is g-contractible. By [1, Theorem 4.6, p. 8], Y is right g-contractible with respect to W. This proves (1).

To show (2)., let $f : W \to W$ be an onto map. Let $i : Y \to W$ be the inclusion map and let $U = W \setminus \{a, \phi(a)\}$. Thus, U is an open subset of W. Now, for each $n \in \mathbb{N}$, let $C_{2n-1} = ad_n \cup D_n \setminus \{a\}$ and let $C_{2n} = \phi(a)\phi(d_n) \setminus \phi(a) \cup \phi(D_n)$. We have that $\{C_n\}_{n\in\mathbb{N}}$ is a sequence in $\text{Comp}(U)$.

Note that $f(i(p)) = p \in U$ and $\liminf C_n = \{p\}$. Thus, (1.1) and (1.2) are satisfied.

From [8, Lemma 2, p. 2180], it follows that there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ in X such that $\lim z_n = p$ and, $z_n \in \overline{NLC(W)}$ and $f(z_n) = d_n$ for each $n \in \mathbb{N}$. Since i is the inclusion map, $i(z_n) = z_n$. Hence, $f(i(z_n)) = f(z_n) = d_n$ and $z_n \in (f \circ i)^{-1}(C_n)$ for each $n \in \mathbb{N}$. Thus $\{p\} \subseteq \liminf(f \circ i)^{-1}(C_n)$. Therefore, $\{p\}$ is a left gR-set of Y with respect to W.

Invoke Theorem 4.9 to show (3). Finally, we shall prove (4). Let $A = [-1, 1] \times \{0\}$ and let $\pi : Y \to A$ be the map defined by $\pi(y, z) = (y, 0)$. Notice that π is retraction and A is contractible. By Corollary 4.2, A is left g-pseudo-contractible with respect to W .

The following example shows that there exist right gR-sets, the condition of being left g-pseudo-contractible with respect to a space does not imply the condition of being right g-pseudo-contractible with respect to the same space and that the converse of [1, Theorem 4.14, p. 10] fails.

Example 4.20. The continua W and Y_H defined in Example 3.6 and Example 3.7 satisfy the following statements:

(1) The continuum W is left g-pseudo-contractible with respect to Y_H.
(2) $\{p\}$ is a right gR-set of W with respect to Y_H.
(3) The continuum W is not right g-pseudo-contractible with respect to Y_H.
(4) There exists a retract A of Y_H such that W is right g-pseudo-contractible with respect to A.

In Example 3.6 we argue that Y_H is g-contractible. By [1, Theorem 4.6, p. 8], W is left g-contractible with respect to Y_H. This proves (1).

In order to show (2), let $f : W \to W$ be an onto map. Let $h : W \to Y_H$ be a retraction and let $U = Y \setminus \{a, \phi(a)\}$. Then, U is an open subset of Y. For each $n \in \mathbb{N}$, let $C_{2n-1} = ad_n \setminus \{a\}$ and $C_{2n} = \phi(a)\phi(d_n) \setminus \{\phi(a)\}$. Thus $\{C_n\}_{n\in\mathbb{N}}$ is a sequence in $\text{Comp}(U)$.

By [3, Lemma 4.15, p. 472], $p \in \limsup f^{-1}(e_n) \cap \limsup f^{-1}(d_n)$. Therefore, $f(p) = p$. Then, $h(f(p)) = p \in U$. Furthermore, $\liminf C_n = \{p\}$. Thus, (r.1) and (r.3) are satisfied.

Now, we shall prove that $\{p\} \subseteq \liminf(h \circ f)^{-1}(C_n)$. Notice that $d_n \in C_n$ for each $n \in \mathbb{N}$. Since h is a retraction, $h(d_n) = d_n$ for each $n \in \mathbb{N}$. By [8, Lemma 2, p. 2180], there exists a convergent sequence $\{z_n\}_{n\in\mathbb{N}}$ to p such that $z_n \in \overline{NLC(W)}$ and $f(z_n) = d_n$ for each $n \in \mathbb{N}$. Hence, $h(f(z_n)) = h(d_n) = d_n \in C_n$. So, $z_n \in (h \circ f)^{-1}(C_n)$. This proves (r.2). Therefore, $\{p\}$ is right gR-set of W with respect to Y_H. By Corollary 4.9, we conclude (3).

Let $A = [-1, 1] \times \{0\}$ and let $\pi : Y_H \to A$ be the map defined by $\pi(y, z) = (y, 0)$. So, the arc A is a retract of Y_H. Apply Corollary 4.2, to conclude that W is right g-pseudo-contractible with respect to A. This prove (4).

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4

Conclusiones

Este proyecto tuvo por resultado dos artículos relacionados al concepto de g-contractibilidad. El primero de ellos es llamado *On g-pseudo-contractibility of continua* y está publicado en la revista *Topology and its Applications*. En este artículo se define el concepto de g-pseudo-contractibilidad el cual generaliza las nociones de g-contractibilidad y pseudo-contractibilidad. Este nuevo concepto se introduce como una herramienta para el estudio de la g-contractibilidad de un continuo. En el desarrollo de este artículo se prueba que dichas nociones son distintas. Además, se puede observar que existen resultados en el estudio de contractibilidad de un continuo que son preservados a el estudio de la g-pseudo-contractibilidad de un continuo. Algunos otros resultados únicamente se garantizan para los continuos contráctiles o para los continuos g-pseudo-contráctiles. En la segunda parte de este artículo se introduce la propiedad de ser g-pseudo-contráctil con respecto, la cual se relaciona con la propiedad de ser g-contráctil con respecto. Se dan propiedades generales de dicho concepto con las cuales es posible determinar cuando un continuo no es g-pseudo-contráctil.

El segundo artículo fue enviado a la revista *Journal of the Korean Mathematical Society* con el nombre de *gR-sets and g-pseudo-contractibility*. Un problema general en topología es hallar condiciones que implican la no contractibilidad de un espacio. El objetivo de este artículo fue dar condiciones necesarias para que un continuo no sea g-pseudo-contráctil y por tanto tampoco sea g-contráctil, pseudo-contráctil o contráctil. Por esta razón se definen los conceptos de gR-conjunto y gR-conjunto con respecto, los cuales son una clase de subconjuntos que obstruyen la g-pseudo-contractibilidad de un continuo.

Este trabajo genera una nueva línea de investigación dentro de la Teoría de Continuos, en la cual aún hay un extenso terreno por explorar de donde se pueden obtener resultados interesantes.

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